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Mini Review

On the Governing Equations for Velocity and Shear Stress of some Magnetohydrodynamic Motions of Rate-type Fluids and their Applications

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Abstract

The governing equations for the shear stress corresponding to some magnetohydrodynamic (MHD) motions of a large class of rate-type fluids are brought to light. In rectangular domains, the governing equations of velocity and shear stress are identical as form. The provided governing equations can be used to solve motion problems of such fluids when shear stress is prescribed on the boundary. For illustration, the motion in an infinite circular cylinder with shear stress on the boundary is discussed.

Introduction

It is well known the fact that exact solutions for different initial-boundary value problems serve a double purpose. Firstly, they can characterize the behavior of a fluid in motion or a solid in deformation, and secondly, they can be used as tests to verify numerical schemes that are developed to study more complex motion or deformation problems. In fluid mechanics, for instance, the study of the fluids' motion leads to a system of partial differential equations with initial conditions in which the fluid velocity, the shear stress or both can be prescribed on the boundary. In the existing literature, there are a lot of exact solutions for motions of the incompressible rate type fluids (Maxwell, Oldroyd-B, Burgers, or generalized Burgers fluids) in which the fluid velocity is given on the boundary. Unfortunately, exact solutions for motions of the same fluids are rare when shear stress is prescribed on the boundary.

However, in many practical situations, what is known is the force applied to the boundary to move it. On the other hand, in Newtonian mechanics force is the cause and kinematics is the effect, and prescribing the shear stress on the boundary is tantamount to give the force applied in order to move it. Early enough, Renardy [1,2] showed that differential expressions of shear stress have to be prescribed on the boundary to formulate well-posed boundary value problems for motions of rate-type fluids. Some exact solutions for such motions of the incompressible generalized Burgers fluids (IGBFs) have been recently provided by Fetecau, et al. [3]. Other interesting results for magnetohydrodynamic (MHD) motions of these fluids have been obtained by Tong [4], Sultan, et al. [5], Sultan and Nazar [6], Abro, et al. [7], Alqahtani and Khan [8] and Hussain, et al. [9]. Nevertheless, as was previously mentioned, more important for us is to find exact solutions for motions of rate-type fluids when shear stress is given on the boundary.

The purpose of this note is to show that the results of the above-mentioned paper can be extended to motions of rate-type fluids in which magnetic effects are taken into consideration.

More exactly, MHD motion problems of these fluids can be solved when the shear stress is prescribed on the boundary. In addition, exact solutions for motions of these fluids with shear stress on the boundary will be easier obtained if similar solutions for the same fluids are known when velocity is prescribed on the boundary and vice versa. Our study refers to isothermal MHD unidirectional motions of IGBFs and incompressible Oldroyd-B fluids (IOBFs) in rectangular and cylindrical domains, respectively. In rectangular domains, the governing equations for the fluid velocity and the non-trivial shear stress are identical as form.

Constitutive and governing equations

The Cauchy stress tensor T and the extra-stress tensor T_E

corresponding to IGBFs satisfy the following constitutive equations [3,9,10].

$$T = -pI + T_E, \quad \left(1 + a_1\frac{D}{D} + a_2\frac{D^2}{Dt^2}\right)T_E = 2\mu\left(1 + a_3\frac{D}{D} + a_4\frac{D^2}{Dt^2}\right)D, \tag{1}$$

where I is the identity tensor, D is the rate of deformation tensor, p is the hydrostatic pressure, μ is the dynamic viscosity of the fluid, D/Dt denotes the time upper-convected derivative, and a_i(i = 1,2,3,4) are dimensional material constants. If $a_4 = 0$, $a_4 = a_2 = 0$ or $a_4 = a_3 = a_2 = 0$ in the second equality (1), the corresponding equations characterize incompressible Burgers, Oldroyd-B, or Maxwell fluids, respectively.

Let us first consider the isothermal unidirectional motion of an IGBF whose velocity field is given by the relation

$$u = u(z,t) = u(z,t)e_y, \tag{2}$$

where e_y is the unit vector along the y-direction of a suitable Cartesian coordinate system x, y, and z. Assuming that the extra-stress tensor T_E, as well as the velocity vector u, is also a function of z and t only, it results that the non-trivial shear stress $\tau(z,t) = T_{EYZ}(z,t)$ and the fluid velocity $u(z,t)$ are related by the next relation [5-10].

$$\left(1 + a_1\frac{\partial}{\partial t} + a_2\frac{\partial^2}{\partial t^2}\right)\tau(z,t) = \mu\left(1 + a_3\frac{\partial}{\partial t} + a_4\frac{\partial^2}{\partial t^2}\right)\frac{\partial u(z,t)}{\partial z}. \tag{3}$$

The balance of linear momentum, in the absence of a pressure gradient in the y-direction but in the presence of a transverse magnetic field of constant magnitude B, reduces to the relevant partial differential equation [8-10].

$$\rho\frac{\partial u(z,t)}{\partial t} = \frac{\partial \tau(z,t)}{\partial z} - \sigma B_0^2 u(z,t), \tag{4}$$

where ρ is the fluid density and σ is its electrical conductivity.

Eliminating the fluid velocity $u(z,t)$ or the shear stress $\tau(z,t)$ between Eqs. (3) and (4) one obtains the governing equations.

$$\rho\left(1 + a_1\frac{\partial}{\partial t} + a_2\frac{\partial^2}{\partial t^2}\right)\frac{\partial u(z,t)}{\partial t} = \mu\left(1 + a_3\frac{\partial}{\partial t} + a_4\frac{\partial^2}{\partial t^2}\right)\frac{\partial^2 u(z,t)}{\partial z^2} - \sigma B^2\left(1 + a_1\frac{\partial}{\partial t} + a_2\frac{\partial^2}{\partial t^2}\right)u(z,t), \tag{5}$$

and

$$\rho\left(1 + a_1\frac{\partial}{\partial t} + a_2\frac{\partial^2}{\partial t^2}\right)\frac{\partial \tau(z,t)}{\partial t} = \mu\left(1 + a_3\frac{\partial}{\partial t} + a_4\frac{\partial^2}{\partial t^2}\right)\frac{\partial^2 \tau(z,t)}{\partial z^2} - \sigma B^2\left(1 + a_1\frac{\partial}{\partial t} + a_2\frac{\partial^2}{\partial t^2}\right)\tau(z,t), \tag{6}$$

For the fluid velocity $u(z,t)$ and the shear stress $\tau(z,t)$, respectively. Consequently, the fluid velocity and the non-trivial shear stress corresponding to such motions of IGBFs satisfy partial differential equations which are identical as form.

Let us now consider the isothermal motion of an IOBF in a cylindrical domain whose velocity field is given by the relation

$$w = w(r,t) = w(r,t)e_z, \tag{7}$$

Where e_z is the unit vector along the z-direction of a convenient cylindrical coordinate system r, θ and z. Supposing that the extra-stress tensor T_E is also a function of r and t only it results that the non-trivial shear stress $\eta(r,t) = T_{Erz}(r,t)$ and the fluid velocity $W(r,t)$ have to satisfy the following partial differential equation [11]

$$\left(1 + a_1\frac{\partial}{\partial t}\right)\eta(r,t) = \mu\left(1 + a_3\frac{\partial}{\partial t}\right)\frac{\partial w(r,t)}{\partial r}. \tag{8}$$

The balance of linear momentum, in the absence of a pressure gradient in the flow direction but in the presence of a transverse magnetic field of constant magnitude B, reduces to the partial differential equation [11]

$$\rho\frac{\partial w(r,t)}{\partial t} = \frac{\partial \eta(r,t)}{\partial r} + \frac{1}{r}\eta(r,t) - \sigma B^2 w(r,t) \tag{9}$$

Eliminating the shear stress $\eta(r,t)$ or the velocity field $W(r,t)$ between Equations (8) and (9) one obtains the following partial differential equations

$$\rho\left(1 + a_1\frac{\partial}{\partial t}\right)\frac{\partial w(r,t)}{\partial t} = \mu\left(1 + a_3\frac{\partial}{\partial t}\right)\left[\frac{\partial^2 w(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial w(r,t)}{\partial r}\right] - \sigma B^2\left(1 + a_1\frac{\partial}{\partial t}\right)w(r,t) \tag{10}$$

respectively,

$$\rho\left(1 + a_1\frac{\partial}{\partial t}\right)\frac{\partial \eta(r,t)}{\partial t} = \mu\left(1 + a_3\frac{\partial}{\partial t}\right)\left[\frac{\partial^2 \eta(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial \eta(r,t)}{\partial r} - \frac{1}{r^2}\eta(r,t)\right] - \sigma B^2\left(1 + a_1\frac{\partial}{\partial t}\right)\eta(r,t) \tag{11}$$

Although the last two equations are not identical to before, the governing equation (11) for shear stress allows us to solve MHD motion problems of IOBFs through an infinite circular cylinder or between two concentric cylinders when shear stress is given on the boundary. It is also worth pointing out the fact that a similar governing equation for the non-trivial shear stress can be provided for MHD motions of the same fluids induced by a couple [12] (rotational shear stress) which acts on the cylindrical boundary.

MHD flow of IOBFs through an infinite circular cylinder

Let us consider the motion of an IOBF through an infinite circular cylinder of radius R that applies an oscillatory shear stress $S\cos(\omega t)$ or $S\sin(\omega t)$ to the fluid. Here S is a constant shear stress and ω is the oscillations' amplitude. The following boundary conditions

$$\eta(R,t) = S\cos(\omega t) \text{ or } \eta(R,t) = S\sin(\omega t) \tag{12}$$

Have to be satisfied if R is the cylinder's radius. Let us denote by $\eta cp(r,t)$ and $\eta SP(r,t)$ the steady state (permanent or long time) solutions of the two motion problems and by

$$\eta_p(r,t) = \eta_{cp}(r,t) + i\eta_{sp}(r,t), \tag{13}$$

The complex shear stress. Here i is the imaginary unit and $\eta_p(r,t)$ has to satisfy the governing equation (11) and the boundary condition

$$\eta_p(R,t) = Se^{i\omega t}. \tag{14}$$

Bearing in mind the linearity of the governing equation (11) and the form of the boundary condition (14), we are looking for a solution of the form

$$\eta_p(r,t) = F(r)e^{i\omega t}. \quad (15)$$

Introducing $\eta_p(r,t)$ from Eq. (15) in (11) and making the change of variable $s = \gamma r$

where

$\gamma = \sqrt{\dfrac{(\sigma B^2 + i\omega\rho)(1+i\omega a_1)}{\mu(1+i\omega a_3)}}$ one attains to a Bessel equation for the function $F(\cdot)$, namely

$$s^2 F''(s) + sF'(s) - (1+s^2)F(s) = 0. \quad (16)$$

Since solutions of Bessel equations are well known in the literature, the steady shear stresses $\eta_{cp}(r,t)$ and $\eta_{sp}(r,t)$ can be easily determined. The corresponding steady-state velocities $W_{cp}(r,t)$ and $W_{sp}(r,t)$ can be also determined using the governing equation (9).

Conclusion

In this short note, it has been provided governing equations for the non-trivial shear stress corresponding to some isothermal MHD motions of the incompressible rate type fluids. In rectangular domains, the governing equations for the fluid velocity and the shear stress are identical as form. This simple remark allows us to easily find exact solutions for MHD motions of rate-type fluids with velocity or shear stress on the boundary if similar solutions for MHD motions of the same fluids are known when shear stress or the fluid velocity is given on the boundary and vice versa. A governing equation for the shear stress has been also brought to light for MHD motions of the same fluids in cylindrical domains. It offers the possibility to study MHD motion problems of the incompressible rate type fluids when shear stress is given on the boundary. Finally, for illustration, a conclusive example has been offered.

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References

1. Renardy M. Recent advances in the mathematical theory of steady flow of viscoelastic fluids. J Nonnewton. Fluid Mech. 1988; 29: 11-24. DOI: 10.1016/0377-0257 (88) 85047-X
2. Renardy M. An alternative approach to inflow boundary conditions for Maxwell fluids in three space dimensions. J Nonnewton Fluid Mech. 1990; 36: 419-425. DOI: 10.1016/0377-0257(90)85022-Q
3. Fetecau C, Rauf A, Qureshi TM, Vieru D. Steady-state solutions for MHD motions of Burgers fluids through porous media with differential expressions of shear on boundary and Applications. Mathematics. 2022; 10(22): 4228. DOI: 10.3390/math 10224228
4. Tong D. Starting solutions for oscillating motions of a generalized Burgers' fluid in cylindrical domains. Acta Mech. 2010; 214; 395-407. DOI: 10.1007/s00707-010-0288-7
5. Sultan Q, Nazar M, Imran M, Ali U. Flow of generalized Burgers fluid between parallel walls induced by rectified sine pulses stress. Bound Value Probl. 2014; 152. DOI: 10.1186/s13661-014-0152-0
6. Sultan Q, Nazar M. Flow of generalized Burgers' fluid between side walls induced by sawtooth pulses stress. J Appl Fluid Mech. 2016; 9: 2195-2204. DOI: 10.18869/acadpub.jafm.68.236.24660
7. Abro KA, Hussain M, Baig MM. Analytical solution of magnetohydrodynamics generalized Burgers' fluid embeded with porosity. Int J Advances Appl Sci. 2017; 4: 80-89. DOI: 10.21833/ijaas.2017.07.012
8. Alqahtani AM, Kha I. Time-dependent MHD flow of a non-Newtonian generalized Burgers' fluid (GBF) over a suddenly moved plate with generalized Darcy's law. Front Phys. 2019; id.214. DOI: 10.3389/fphy.2019.00214
9. Hussain M, Quayyum M, Sidra A. Modeling and analysis of MHD oscillatory flows of generalized Burgers' fluid in a porous medium using Fourier transform. J Math. 2022; 2373084. DOI: 10.1155/2022/2373084
10. Fetecau C, Akhtar S, Morosanu C, Porous and magnetic effects on modified Stokes' problems for generalized Burgers fluids. Dynamics. 2023; 3: 803-819. DOI: 10.3390/dynamics3040044
11. Hamza SEE. MHD flow of an Oldroyd-B fluid through porous medium in a circular channel under the effect of time-dependent gradient. *Am J Fluid Dyn*. 2017; 7: 1-11. DOI: 10.5923/j.ajfd.20170701.01
12. Bandelli R, Rajagopal KR. Start-up flows of second-grade fluids in domains with one finite dimension. Int J Non-Linear Mech. 1995; 30: 817-839. DOI: 10.1016/0020-7462(95)00035-6

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